Lifespan Persistence Through Numerical Analysis Of Initial Value Problems

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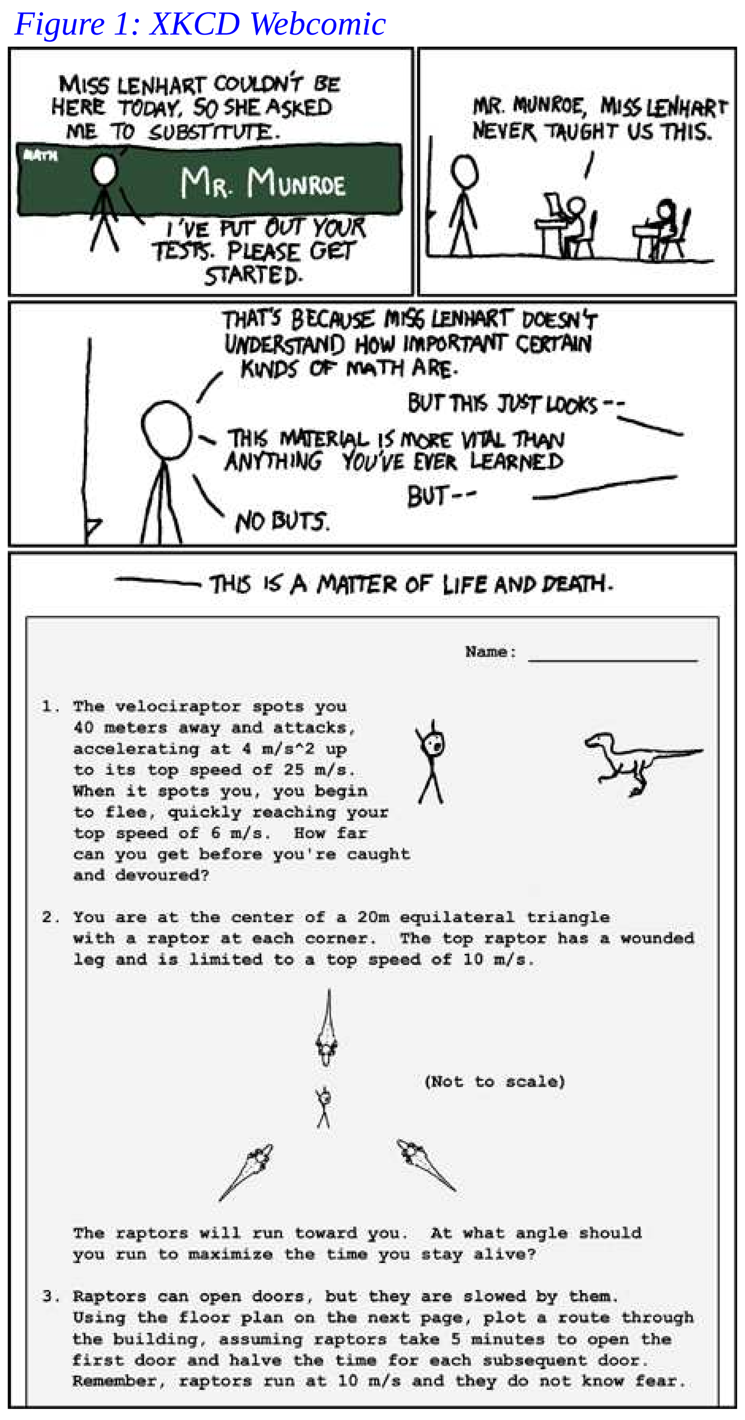
Figure 1: XKCD Webcomic



Introduction

The development of iterative methods for finding approximate solutions of systems of nonlinear equations is a fundamental subject in computational mathematics and evolutionary game theory.

See *Figure 1 - XKCD Webcomic*. Problem number two prompts a classic discussion of solutions of systems of nonlinear equations. Those who study mathematics will attempt to find a closed-form differential representation of the objects with respect to time, then attempt to find a general solution to the differential equations.

However, many nonlinear systems cannot be explicitly solved. Instead, methods for finding approximate solutions exist, but require intense computation in some cases. The scope of this study will cover the computation of these numeric methods for finding these approximate solutions.

Clarifications

The following clarifications were contributed by Matthew Beckler, University of Minnesota, following his consultation with Randall Munroe, author of the webcomic:

- *The human will run radially outward in a straight line.*
  This will allow us to characterize a run path with just the angle of the run.
- *The velociraptors need to be very close (10 cm) to your position to eat you.*
  The eating distance really only slightly effects the survival time.
- *The human has infinite acceleration to 6 m/s.*
  This really would only effect the time slightly. The shape of the solution would not change significantly.
- *The velociraptors can turn any angle at any velocity.*
  This has a large effect on the solution, and was only implemented after consulting with the author, who said, *"I actually remember seeing some research saying that Utahraptor may have sacrificed some speed in favor of quick turning ability, so there's some justification for that one. So I would say you let them turn as fast as they want, but they always run toward your current position (in reality, raptors are far more cunning than this)."*

Analysis

Because the direction of each raptor changes with every movement of the human, an exact solution is difficult to obtain. An approximate model can be simulated by the computer.

The angle of maximal survival time is found using an iterative method. A *trial* simulates nine angles in an interval and retains the most successful angle. The next *trial* analyses a smaller interval around that angle. This process continues for ten *trials*.

Simulations consist of a series of discrete *time steps*, imitating the continuous nature of time. Each *time step* updates the position of each raptor and human. If the *time step* is too small, the simulation will take too long to run. If it is too large, the results will not be accurate. For this problem, the *time step* is 0.0001 seconds.

Solution

By observation, it is inferred that two solutions exist, symmetric about the y-axis. Therefore, finding only one solution in either Quadrants I and IV or Quadrants II and III is sufficient. The other solution is reflected about the y-axis.
Trigonometric computation is simplest in Quadrants I and IV, so this will be the initial interval to explore.
See *Figure 2: [-π/2, π/2]*.

$Trial_1$ processes nine angles in the closed interval $[-\pi/2, \pi/2]$, with uniform angle increment $\pi/8$. The most successful angle, $theta_1$, is found to be $-\pi/2$.
$Trial_2$'s interval will be $[theta_1-\pi/8, theta_1+\pi/8]$ with increment $(\pi/8)/4$.

$Trial_2$ processes nine angles in the closed interval $[-5\pi/8, -3\pi/8]$, with uniform angle increment $\pi/32$. The most successful angle, $theta_2$, is found to be $-\pi/2$.
$Trial_3$'s interval will be $[theta_2-\pi/32, theta_2+\pi/32]$ with increment $(\pi/32)/4$.

$Trial_3$ processes nine angles in the interval $[-17\pi/32, -15\pi/32]$, with uniform angle increment $\pi/128$. The most successful angle, $theta_3$, is found to be $-\pi/2$.
$Trial_4$'s interval will be $[theta_3-\pi/128, theta_3+\pi/128]$ with increment $(\pi/128)/4$.

This process continues for ten trials. $Theta_{10} = -1.570796$. $Time_{10} = 3.0301$ s.
$Raptor_{1\ velocity} = 10.0$ m/s. $Raptor_{2\ velocity} = Raptor_{3\ velocity} = 12.1204$ m/s.

However, it is immediately obvious that this solution is incorrect!
Logically, one would run toward the wounded raptor in Quadrant I.
The initial interval is changed to $[0, \pi/2]$ to reflect this assumption.
See *Figure 3: [0, π/2]*.

$Trial_1$ processes nine angles in the closed interval $[0, \pi/2]$, with uniform angle increment $\pi/16$. The most successful angle, $theta_1$, is found to be $3\pi/16$.
$Trial_2$'s interval will be $[theta_1-\pi/16, theta_1+\pi/16]$ with increment $(\pi/16)/4$.

$Trial_2$ processes nine angles in the closed interval $[\pi/8, \pi/4]$, with uniform angle increment $\pi/64$. The most successful angle, $theta_2$, is found to be $11\pi/64$.
$Trial_3$'s interval will be $[theta_2-\pi/64, theta_2+\pi/64]$ with increment $(\pi/64)/4$.

$Trial_3$ processes nine angles in the closed interval $[5\pi/32, 3\pi/16]$, with uniform angle increment $\pi/256$. The most successful angle, $theta_3$, is found to be $23\pi/128$.
$Trial_4$'s interval will be $[theta_3-\pi/256, theta_3+\pi/256]$ with increment $(\pi/256)/4$.

This process continues for ten trials. $Theta_{10} = 0.569454$. $Time = 3.0806$ s.
$Raptor_{1\ velocity} = 10.0$ m/s. $Raptor_{2\ velocity} = Raptor_{3\ velocity} = 12.3224$ m/s.

Why did the first initial interval provide an incorrect solution?

See *Figure 4: Angle vs. Lifespan*.
This "radar" shows survival time given the angle one chooses to run. Because the human's velocity is constant, units of time and distance are interchangeable in the graph. The red line is a "boundary," showing the maximum distance achievable in any given direction.

Each raptor assumed the same initial acceleration, and $Raptor_1$'s immobility did not have any affect until *2.5 seconds* into the simulation. After *2.5 seconds*, the difference is only slightly noticeable, but it **does** matter for the theory of iterative methods. The difference is visible in *Figure 4*, as the boundary is further away from the human in Quadrants I and II than on the negative y-axis.

It is clear why the first initial interval provided an incorrect solution. Although the correct solution was in Quadrant I, the angle increment was too large and caused $trial_1$ to omit the solution.

Now that the maximal survival angle is known, the full run for all raptors and the human can be plotted. See *Figure 5: Maximum Lifespan*.
This shows the path each raptor ran to overtake you.

Figure 2: [-π/2, π/2]

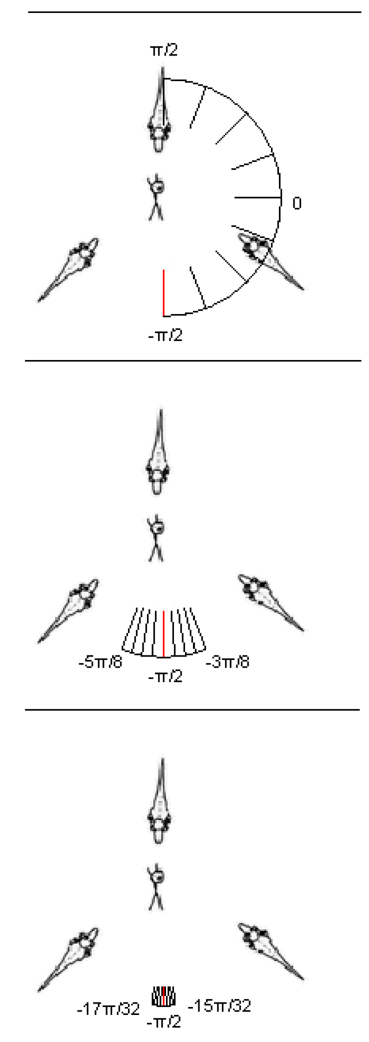


Figure 3: [0, π/2]

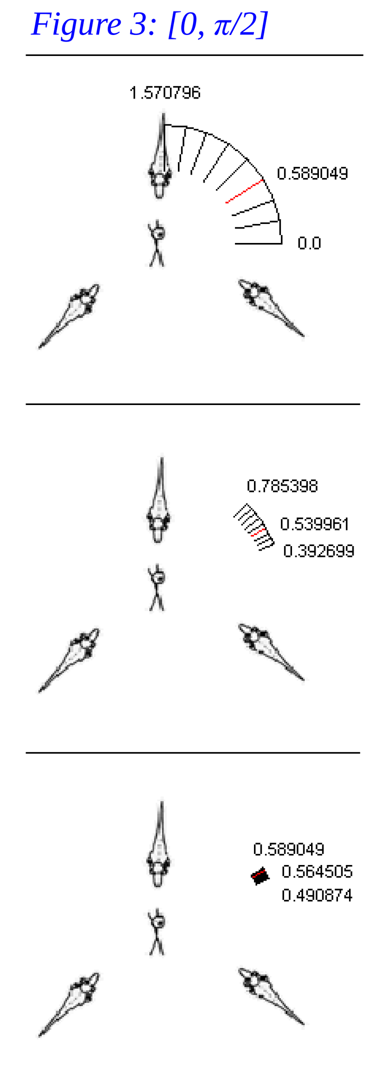


Figure 4: Angle vs. Lifespan

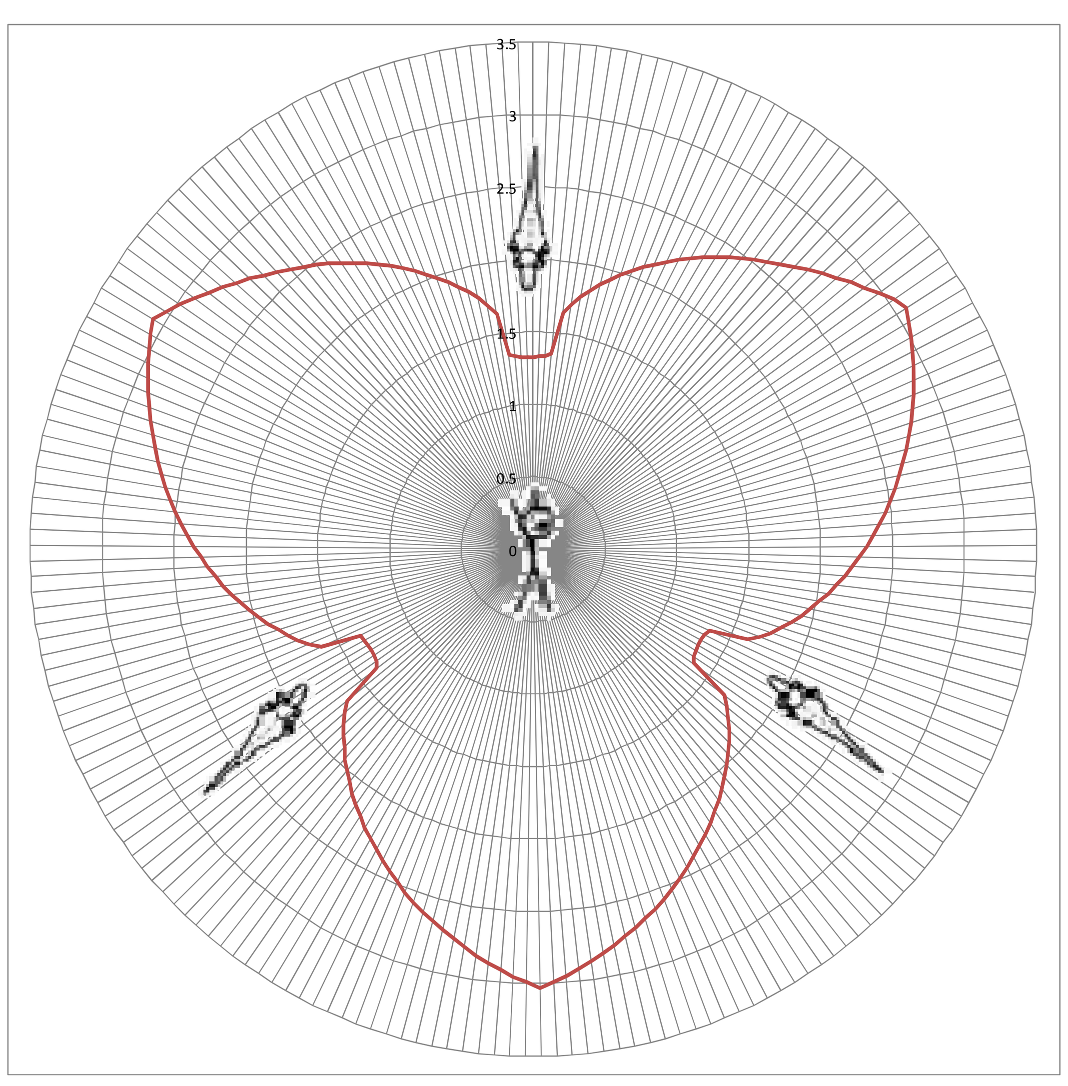
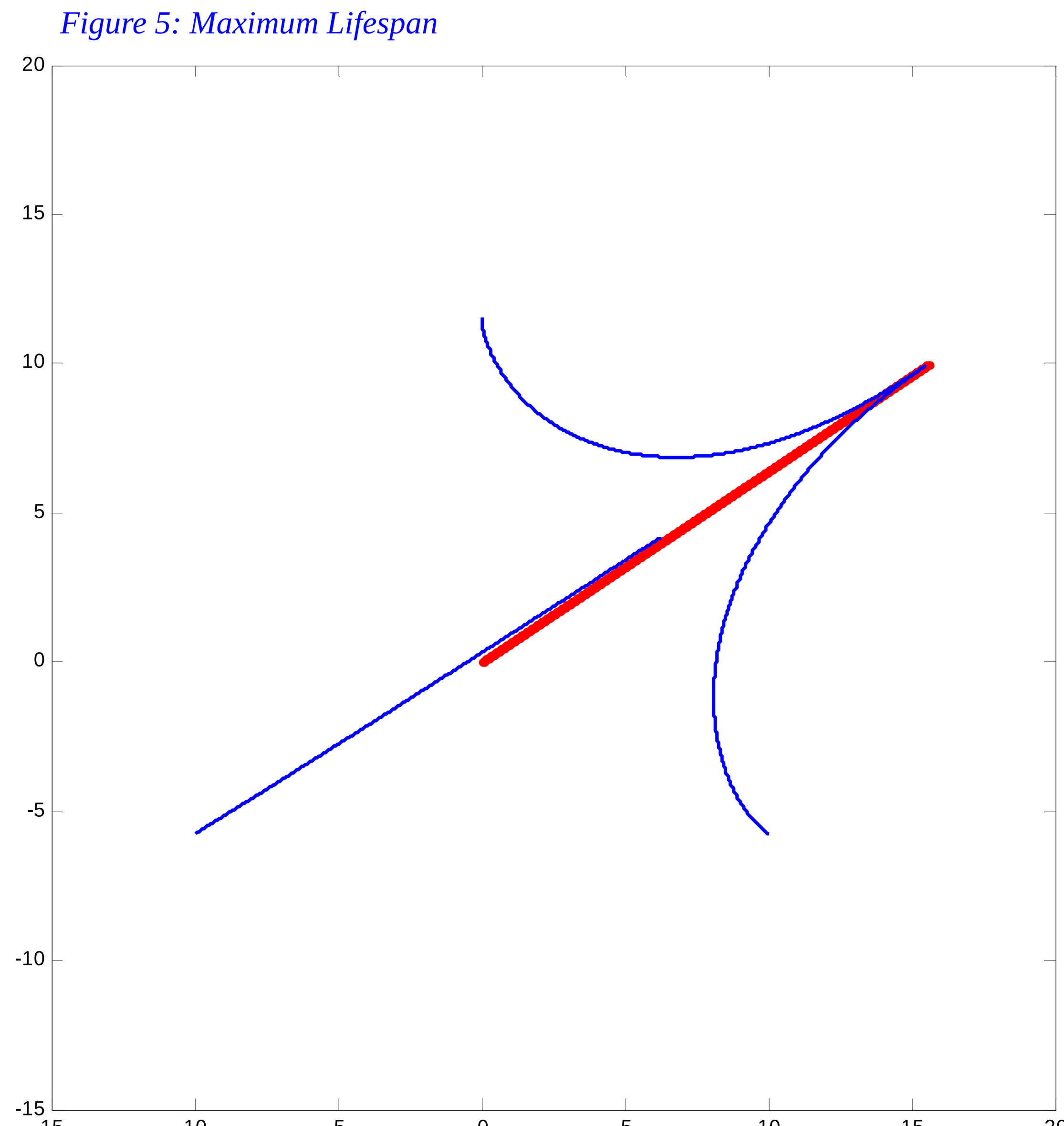


Figure 5: Maximum Lifespan



Remarks

Changing the Initial Conditions
Modifying the initial conditions may or may not affect the shape of the solution. This depends on whether or not the initial interval contains prominent values such as the maxima, minima, and points of inflection. Regardless, it is important to note the difference in how the solution is computed. Some initial conditions will allow the solution to be found more efficiently than other initial conditions. Also, in some cases the correct solution will not be found, as described in *Figure 2*.

Alternate Methods
It can also be observed that at the end of the most successful run, two of the raptors will be equally distant from the human, and the position of the third raptor can be ignored. Based on these assumptions, many variants of the iteration method can be developed.
One variant only calculates the positions of the two raptors closest to the human's run angle.
Another variant of the method involves only one initial angle, instead of an interval of angles. The single run is simulated, and the angle is then incremented or decremented a certain amount. The calculation of this increment will involve the distance from the human to the second raptor. The iteration is repeated until two raptors are equally distant from the human within the tolerance.
However, this method suffers from the same drawbacks as seen in *Figure 2*. It is nearly impossible to calculate the correct solution to this problem with only one initial guess, given that multiple local maxima exist.

Exact Solutions
The lines tangent to the paths of the three raptors at any time t intersect at the position of the human at time t. This observation could be used to create a system of differential equations with which the solution could be explicitly solved. However, the derivation of these equations and the methodology of finding a solution of the system is far beyond the scope of this study, which has focused on the numerical approximation of the solution.

Special Thanks to Randall Munroe, author of xkcd A WEBCOMIC OF ROMANCE, SARCASM, MATH, AND LANGUAGE.

Also, Special Thanks to Matthew Beckler and Dr. William Layton.